2.4 A Theorem About Monte Carlo Approximation of Products of Means

Let $X_{j1}, \ldots, X_{jm}$, $j = 1, \ldots, J$ be $J \geq 2$ sequences of independent random variables, each of length m. For each j, $X_{j1}, \ldots, X_{jm}$ are identically and independently distributed with $\mathbb{E}X_{ji} = \mu_j$. Suppose that we are interested in estimating $\prod_{j=1}^{J} \mu_j$. Then the two estimators

$$\bar{\mu} = \prod_{j=1}^{J} \frac{1}{m} \sum_{i=1}^{m} X_{ji} \qquad \text{and} \qquad \tilde{\mu} = \frac{1}{m} \sum_{i=1}^{m} \prod_{j=1}^{J} X_{ji}$$

are both unbiased for $\prod_{j=1}^{J} \mu_j$, but $\mathrm{Var}(\bar{\mu}) \leq \mathrm{Var}(\tilde{\mu})$, with strict inequality holding when the X_{ji} are non-degenerate random variables for at least two $j \in \{1, \ldots, J\}$.

Proof: Unbiasedness is straightforward to show:

$$\mathbb{E}(\bar{\mu}) = \mathbb{E}\left(\prod_{j=1}^{J} \frac{1}{m} \sum_{i=1}^{m} X_{ji}\right) = \prod_{j=1}^{J} \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}X_{ji} = \prod_{j=1}^{J} \mu_j \tag{2.11}$$

and

$$\mathbb{E}(\tilde{\mu}) = \mathbb{E}\left(\frac{1}{m} \sum_{i=1}^{m} \prod_{j=1}^{J} X_{ji}\right) = \frac{1}{m} \sum_{i=1}^{m} \prod_{j=1}^{J} \mathbb{E}X_{ji} = \prod_{j=1}^{J} \mu_j. \tag{2.12}$$

To show that $\mathrm{Var}(\bar{\mu}) \leq \mathrm{Var}(\tilde{\mu})$, it will suffice to show that $\mathbb{E}\bar{\mu}^2 \leq \mathbb{E}\tilde{\mu}^2$, because the squares of the expected values of each estimator will be equal and $\mathrm{Var}(X) = \mathbb{E}X^2 - [\mathbb{E}X]^2$. We start by simplifying the expression for $\mathbb{E}\bar{\mu}^2$.

$$\begin{aligned}
\mathbb{E}\bar{\mu}^2 &= \mathbb{E}\left(\prod_{j=1}^{J} \frac{1}{m} \sum_{i=1}^{m} X_{ji}\right)^2 \\
&= \mathbb{E}\left(\prod_{j=1}^{J} \frac{1}{m^2} \left(\sum_{i=1}^{m} X_{ji}\right)^2\right) \\
&= \prod_{j=1}^{J} \frac{1}{m^2} \mathbb{E}\left(\sum_{i=1}^{m} X_{ji}^2 + 2\sum_{i=1}^{m} \sum_{k<i} X_{ji} X_{jk}\right) \\
&= \prod_{j=1}^{J} \frac{1}{m^2} \left(\sum_{i=1}^{m} \mathbb{E}X_{ji}^2 + 2\sum_{i=1}^{m} \sum_{k<i} \mu_j^2\right) \\
&= \prod_{j=1}^{J} \frac{1}{m^2} \left(m\mathbb{E}X_{ji}^2 + m(m-1)\mu_j^2\right) \\
&= \prod_{j=1}^{J} \frac{1}{m} \left(\mathbb{E}X_{ji}^2 + (m-1)[\mathbb{E}X_{ji}^2 - \mathrm{Var}(X_{ji})]\right)
\end{aligned}$$

$$= \prod_{j=1}^{J}\Big(\mathbb{E}X_{ji}^2 - \frac{(m-1)}{m}\mathrm{Var}(X_{ji})\Big).$$

We know that $\mathrm{Var}(X_{ji}) \leq \mathbb{E}X_{ji}^2$, so we may write $\mathrm{Var}(X_{ji}) = \delta_j \mathbb{E}X_{ji}^2$, where $0 \leq \delta_j \leq 1$. Doing so, we may now rewrite $\mathbb{E}\bar{\mu}^2$.

$$\begin{aligned}
\mathbb{E}\bar{\mu}^2 &= \prod_{j=1}^{J}\Big(\mathbb{E}X_{ji}^2 - \frac{(m-1)\delta_j}{m}\mathbb{E}X_{ji}^2\Big) \\
&= \prod_{j=1}^{J}\Big([(1-\delta_j) + \delta_j/m]\mathbb{E}X_{ji}^2\Big) \\
&= \prod_{j=1}^{J}\Big[(1-\delta_j) + \delta_j/m\Big]\prod_{j=1}^{J}\mathbb{E}X_{ji}^2. \qquad (2.13)
\end{aligned}$$

In similar fashion, we simplify the expression for $\mathbb{E}\tilde{\mu}^2$:

$$\begin{aligned}
\mathbb{E}\tilde{\mu}^2 &= \mathbb{E}\Big(\frac{1}{m}\sum_{i=1}^{m}\prod_{j=1}^{J}X_{ji}\Big)^2 \\
&= \frac{1}{m^2}\mathbb{E}\Big[\sum_{i=1}^{m}\Big(\prod_{j=1}^{J}X_{ji}\Big)^2 + 2\sum_{i=1}^{m}\sum_{k<i}\Big(\prod_{j=1}^{J}X_{ji}X_{jk}\Big)\Big] \\
&= \frac{1}{m^2}\mathbb{E}\Big[\sum_{i=1}^{m}\prod_{j=1}^{J}X_{ji}^2 + 2\sum_{i=1}^{m}\sum_{k<i}\Big(\prod_{j=1}^{J}\mu_j^2\Big)\Big] \\
&= \frac{1}{m^2}\Big(\sum_{i=1}^{m}\prod_{j=1}^{J}\mathbb{E}X_{ji}^2 + m(m-1)\prod_{j=1}^{J}\mu_j^2\Big) \\
&= \frac{1}{m}\Big(\prod_{j=1}^{J}\mathbb{E}X_{ji}^2 + (m-1)\prod_{j=1}^{J}[\mathbb{E}X_{ji}^2 - \mathrm{Var}(X_{ji})]\Big) \\
&= \frac{1}{m}\Big(\prod_{j=1}^{J}\mathbb{E}X_{ji}^2 + (m-1)\prod_{j=1}^{J}[\mathbb{E}X_{ji}^2 - \delta_j\mathbb{E}X_{ji}^2]\Big) \\
&= \Big[\frac{1}{m} + \frac{(m-1)}{m}\prod_{j=1}^{J}(1-\delta_j)\Big]\prod_{j=1}^{J}\mathbb{E}X_{ji}^2. \qquad (2.14)
\end{aligned}$$

And so, inspecting (2.13) and (2.14) it is clear that $\mathrm{Var}(\bar{\mu}) \leq \mathrm{Var}(\tilde{\mu})$ if and only if

$$\prod_{j=1}^{J}\Big[(1-\delta_j) + \delta_j/m\Big] \leq \frac{1}{m} + \frac{(m-1)}{m}\prod_{j=1}^{J}(1-\delta_j), \qquad (2.15)$$

an inequality which may be verified as follows: note that $\prod_{j=1}^{J}[(1-\delta)+\delta]=1$, but the product may be written as a sum

$$\prod_{j=1}^{J}[(1-\delta_j)+\delta_j]=\prod_{j=1}^{J}(1-\delta_j)+\phi=1$$

where ϕ is a sum of 2^J-1 terms, each of the form

$$\prod_{j\in A}\delta_j\prod_{j\in A^c}(1-\delta_j)$$

where A is a non-empty subset of $\{1,\ldots,J\}$, and A^c is its complement. Observe then,

$$\begin{aligned}
\prod_{j=1}^{J}(1-\delta_j)+\phi &= 1\\
\frac{1}{m}\prod_{j=1}^{J}(1-\delta_j)+\frac{\phi}{m} &= \frac{1}{m}\\
\frac{\phi}{m} &= \frac{1}{m}-\frac{1}{m}\prod_{j=1}^{J}(1-\delta_j)\\
\prod_{j=1}^{J}(1-\delta_j)+\frac{\phi}{m} &= \frac{1}{m}+\frac{m-1}{m}\prod_{j=1}^{J}(1-\delta_j). \qquad (2.16)
\end{aligned}$$

The right side of (2.16) is the same as the right side of (2.15), so to prove the inequality in (2.15), we need merely demonstrate that

$$\prod_{j=1}^{J}\Big[(1-\delta_j)+\delta_j/m\Big]\leq\prod_{j=1}^{J}(1-\delta_j)+\frac{\phi}{m}.$$

This may be done by noting that the left side expands into

$$\prod_{j=1}^{J}(1-\delta_j)+\varphi$$

where φ is a sum of 2^J-1 terms, each of which has the form

$$\prod_{j\in A}\frac{\delta_j}{m^z}\prod_{j\in A^c}(1-\delta_j)$$

where A and A^c are as above and z is the number of elements in the set A. (Since A is non-empty, $1\leq z\leq J$.) For $m>1$, each such term is clearly less than or equal to the corresponding term in the sum for ϕ/m so $\varphi\leq\phi/m$. Transparently, the equality holds only when all but one of the δ_j are zero. QED.